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Robust Ordinal Regression for Dominance-based Rough Set Approach to multiple criteria sorting

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ABSTRACT

We present a new multiple criteria sorting method deriving from Dominance-based Rough Set Approach (DRSA). The preference information supplied by the Decision Maker (DM) is a set of possibly imprecise and inconsistent assignment examples on a subset of reference alternatives relatively well-known to the DM. To structure the data we use DRSA, and subsequently, represent the assignment examples by all minimal sets of rules covering all alternatives from the lower approximations of class unions. Such a set of rules is called minimal-cover set – it is one of the instances of the preference model compatible with DM's preference information. In this way, we implement the principle of Robust Ordinal Regression (ROR) to decision rule preference model. For each alternative, we derive the necessary and possible assignments specifying the range of classes to which the alternative is assigned by all or at least one compatible set of rules, respectively, as well as class acceptability indices. We also introduce the notion of a representative compatible minimal-cover set of rules whose selection builds on the results of ROR, addressing the robustness concern. Application of the approach is demonstrated by classifying 69 land zones in 4 classes representing different risk levels.

1. Introduction

In the multiple criteria sorting problem, alternatives need to be assigned to one or more classes, based on their evaluations on multiple and potentially conflicting criteria [51]. The classes considered here are defined before the assignment procedure is run, which means that, unlike clusters, they do not result from the analysis. Moreover, one imposes a preference relation between the classes, which implies that, contrary to the nominal classification problems, these classes are completely ordered. Sorting is among the most frequent real-world decision problems in such various fields as medicine, finance, environmental protection, tourism, and risk assessment. For example, patients need to be assigned to classes indicating the degree of severity of their illness based on several symptoms [45,46,50]. Job applicants may be assigned to elementary, standard, master, or expert level professionals based on their skills, qualifications, and professional experience [38]. Customer satisfaction is investigated to facilitate the development of a marketing strategy [35]. The level of risk associated

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with nanomaterials often needs to be assessed with respect to their toxicity, physico-chemical characteristics, and the expected environmental impacts through the product life cycle [44]. Finally, dividing alternatives into sorted groups is often used in the recommender systems [36].

Arriving at a final assignment requires aggregating vector evaluations of alternatives in a way consistent with a value system of the DM. For this reason, we need to employ a preference model, whose proper exploitation results in the recommendation proposed to the DM. There are three main preference models considered in Multiple Criteria Decision Aiding (MCDA): utility (or value) function [33], the outranking approach [12,37], and the set of decision rules [15,17,39]. In the traditional aggregation paradigm used in MCDA, the criteria aggregation model is known a priori, while the comprehensive preference is unknown. On the contrary, the disaggregation approaches construct preference models from the given comprehensive preferences by applying some inference procedures [24]. In the context of multiple criteria sorting problems, the philosophy underlying these approaches is to find compatible instances of the preference model (utility functions, outranking relations, or sets of rules) which are able to reproduce exemplary statements provided by the DM (e.g., assignment examples concerning some reference alternatives relatively well-known to the DM). Such disaggregation approaches are considered more interesting, because they require less cognitive effort from the DM in answering questions concerning her/his preferences.

When it comes to decision rules, one of the prevailing methods used to structure the data prior to the induction procedure is Dominance-based Rought Set Approach (DRSA) (see, e.g., [3,8,16,40]). DRSA is based on the substitution of the indiscernibility relation by a dominance relation in the rough approximation of decision classes. Then, different rule induction strategies may be applied for representing the knowledge concerning these approximations and for subsequent assignment of non-reference alternatives to decision classes using these rules. Some induction strategies offer a minimal set of rules representing the assignment examples in the most concise way (i.e., without any redundant statements) [7,19]. This strategy is called minimal-cover strategy. It is usually performed by greedy heuristics of sequential covering type, giving an approximately-minimal-cover set of rules. Other strategies aim at discovering a set of decision rules satisfying some pre-defined user's requirements, e.g., with respect to minimal support, maximal length or maximal number of rules [42]. Yet other strategies propose to induce an exhaustive set of rules [5,11], i.e. all rules compatible with the provided preference information, which form the most comprehensive knowledge base with respect to the analyzed data set. Finally, the recent trend consists in integrating several base classifiers into ensembles or committees of classifiers [34,47]. Several methods have been proposed to get diverse base classifiers to be integrated within an ensemble of classifiers, e.g., by changing the distribution of examples in the learning set, manipulating the input features, and different learning algorithms to the same data. The best known methods are bagging [6] and boosting [10] which modify the set of objects by sampling or weighting particular objects and use the same learning algorithm to create base classifiers.

The above review proves that among many sets of rules reproducing the provided preference information, only one specific set of rules or a small subset of these sets has been traditionally used to work out a recommendation. Such a choice is either arbitrary or requires involvement of the DMs, which is not easy for most of them, in particular, if their expectations need to be specified prior to the rule induction. This inconvenience has been already noticed in case of other preference models used in MCDA. With the aim of taking into account all compatible instances of a preference model, and not only one such instance or a limited set of such instances, one has proposed the principle of Robust Ordinal Regression [9,21]. It has been first introduced for value functions [14,22], and then adapted for outranking relations [13,32].

The aim of this paper is to adapt Robust Ordinal Regression to the rule-based preference model, and to take into account all minimal-cover (MC) sets of decision rules compatible with the provided assignment examples when computing recommendation for the non-reference alternatives. An MC set of rules is a single base instance of the preference model. In order to generate all MC set of rules, first an exhaustive set of rules needs to be constructed. Subsequently, all minimal sets of rules are generated by exploiting the set of all rules. This approach requires a considerable amount of computational time and operational memory, as the complexity of generating all rules is exponential and finding the minimal set of rules is proved to be NP-hard. The complexity of these procedures has been already discussed in the literature several times [23,41], and we are conscious of this, however in case of preference information, the set of assignment examples is often rather small (below 100) and the complexity of this approach is not a limiting factor. Thus, we wish to skip this discussion, and focus first on presenting the technique for obtaining all compatible MC sets of rules, and then applying them on the set of all alternatives.

Note that although compatible sets of rules reproduce the assignment examples provided by the DM, the assignment of non-reference alternatives that can be obtained for any of these sets of rules can vary significantly. We investigate the diversity of the recommendation suggested by different sets of rules by producing two types of assignment for each alternative from the considered set of alternatives. Precisely, a is *necessarily* assigned to Cl_h, if and only if a is assigned to Cl_h for all sets of rules compatible with the provided preference information, and a is *possibly* assigned to Cl_h, if and only if a is assigned to Cl_h for at least one compatible set of rules. Since all compatible MC sets of rules need to be induced, we are also able to compute class acceptability indices defined as the share of compatible sets of rules assigning an alternative to a single class or a set of contiguous classes. Then, we continue the robustness analysis in a twofold way. First, we select a representative MC set of rules which builds on the class acceptability indices and highlights the most stable part of the robust sorting. In this way, it enables a synthetic representation of the results of ROR, and thus, its representativeness should be interpreted in the sense of robustness preoccupation. Finally, for each alternative, we indicate rules which are decisive in terms of its recommendation for the greatest number of compatible MC sets of rules. These rules directly influence the assignment suggested by different instances, being of a particular interest to the DM willing to understand the obtained recommendation.

We continue by introducing the notation and recalling some basic concepts of DRSA in Section 2. We present algorithms for generating all compatible rules and all compatible minimal sets of rules in Sections 3 and 4, respectively. Section 5 is devoted to a sorting method that we employ. We continue with the discussion on the possible and necessary assignments in Section 6. Procedure for selection of a representative MC set of rules is presented in Section 7, while the notion of decisive rules is introduced in Section 8. To illustrate the introduced concepts and algorithms, in different sections we refer to a real-world multiple criteria environmental problem. Section 9 concludes the paper.

2. Notation and basic concepts

We shall use the following notation:

- $A = \{a_1, a_2, \ldots, a_i, \ldots, a_n\}$ – a finite set of n alternatives;
- $A^R = \{a^*, b^*, \ldots\}$ – a finite set of reference alternatives, on which the DM accepts to express holistic preferences; we assume that $A^R \subseteq A$;
- $G = \{g_1, g_2, \ldots, g_j, \ldots, g_m\}$ – a finite set of m evaluation criteria, $g_j : A \rightarrow \mathbb{R}$ for all $j \in J = \{1, 2, \ldots, m\}$; although number-coded they may have ordinal or cardinal scales; without loss of generality, we assume that the greater the evaluation, the better;
- $X_j = \{x_j \in \mathbb{R} : g_j(a_i) = x_j, a_i \in A\}$ – the set of all different evaluations on $g_j, j \in J$;
- $Cl_1, Cl_2, \ldots, Cl_p$ – p predefined preference ordered classes having a semantic definition, where Cl_{h+1} is preferred to $Cl_h, h = 1, \ldots, p-1$, moreover, $H = \{1, \ldots, p\}$;
- $Cl_t^{\geqslant} = \bigcup_{s \geqslant t} Cl_s$ – upward union of classes, $t = 2, \ldots, p$;
- $Cl_t^{\leqslant} = \bigcup_{s \leqslant t} Cl_s$ – downward union of classes, $t = 1, \ldots, p-1$.

Dominance relation. The sole information coming out from the analysis of the performances of the alternatives on multiple criteria is the dominance relation. Let us recall that alternative $a \in A$ *weakly dominates* alternative $b \in A$ with respect to a set of criteria $P \subseteq G$ ($a \Delta_P b$) if and only if $g_j(a) \geqslant g_j(b)$ for all $g_j \in P$ (i.e., a is at least good as b on all criteria $g_j \in P$).

Assignment examples. We assume that the DM provides a set of assignment examples, each one consisting of a reference alternative $a^* \in A^R \subseteq A$ and its desired assignment:

$$a^* \rightarrow [Cl_{L^{DM}(a^*)}, Cl_{R^{DM}(a^*)}], \tag{1}$$

where $[Cl_{L^{DM}(a^*)}, Cl_{R^{DM}(a^*)}]$ is an interval of contiguous classes $Cl_{L^{DM}(a^*)}, Cl_{L^{DM}(a^*)+1}, \ldots, Cl_{R^{DM}(a^*)}$. An assignment example is said to be precise if $L^{DM}(a^*) = R^{DM}(a^*) = h$ for some $h \in H$, and imprecise, otherwise.

Illustrative example (part 1): input data and assignment examples

For the purpose of illustration, we re-analyze a real world risk assessment problem of zoning the watershed of Moulinet and Violettes (Low Normandy, France), which was originally presented in [1]. We consider a set of 69 land zones evaluated on five criteria with decreasing direction of preference:

- g_1: the overall slope of the land zone;
- g_2: the quality of the connectivity between the land zone and the stream;
- g_3: the type of embankment in the lower part of the land zone;
- g_4: the nature of the crop on the soil of the land zone;
- g_5: the bank alteration by the cows, when they drink water directly from the stream.

The performances of zones on the five criteria are presented in Table 1. The objective of the study is to indicate the most appropriate intervention for protecting the reproduction habitat of the salmonid fishes in these watersheds. The interventions are classified according to the risk level, as follows: very high risk (Cl_1), high risk (Cl_2), intermediate risk (Cl_3), and low or no risk (Cl_4). Thus, the objective of this study boils down to assignment of each zone to one of the classes of intervention. The preference information consists of the exemplary assignments for the 23 reference zones (these are distinguished with (Ref. = Y) in Table 1). There are 7, 9, 5, and 2 zones assigned by the DM to classes Cl_4, Cl_3, Cl_2, and Cl_1, respectively. The other 46 zones have an unknown assignment.

Granules of knowledge. The set of reference alternatives dominating $a \in A^R, D_P^+(a)$, is called the P-dominating set:

$$D_P^+(a) = \{b \in A^R, b \Delta_P a\}. \tag{2}$$

The set of reference alternatives dominated by $a \in A^R$, $D_P^-(a)$, is called the P-dominated set:

$$D_P^-(a) = \{b \in A^R, a \Delta_P b\}. \tag{3}$$

Rough approximations. The granules of knowledge induced by the dominance relation Δ_P are used for rough approximation of upward and downward unions of classes. The P-lower and P-upper approximations of $Cl_h^{\geqslant}$, $h = 2, \ldots, p$, with respect to $P \subseteq G$, are defined, respectively, as:

Table 1
Land zones' performances (names as in [1]), their inclusion in the reference set A^R (Ref. = Y), and original opinion of environmental experts (*Exp.* = class index).

Zone	Ref.	g_1	g_2	g_3	g_4	g_5	*Exp.*
a_1	Y	1.0	2	4	3	1	4
a_2	N	10.1	2	1	1	1	
a_3	Y	8.3	2	6	1	1	4
a_5	N	219.5	3	4	4	1	
a_6	Y	49.9	7	6	4	2	2
a_7	Y	208.9	7	6	1	8	1
a_8	N	67.7	2	1	4	1	
a_9	N	141.1	3	4	4	1	
a_{12}	Y	91.6	1	4	4	1	3
a_{14}	N	44.8	1	4	3	1	
a_{15}	Y	8.3	1	4	3	1	4
a_{16}	N	14.8	1	6	1	1	
a_{17}	N	53.8	1	1	1	1	
a_{20}	N	289.2	1	4	3	1	
a_{21}	N	66.4	1	6	1	1	
a_{22}	N	128.5	5	4	1	1	
a_{24}	N	55.0	3	4	1	1	
a_{25}	N	135.4	7	6	1	2	
a_{26}	N	161.4	7	6	1	2	
a_{27}	Y	163.1	7	6	1	1	2
a_{28}	N	244.8	5	4	4	1	
a_{29}	N	215.1	3	4	4	1	
a_{30}	Y	49.8	3	4	6	1	3
a_{31}	Y	66.4	3	1	1	1	4
a_{32}	Y	150.4	3	1	6	1	3
a_{33}	Y	63.0	2	6	6	1	3
a_{35}	N	33.4	2	6	1	1	
a_{36}	N	30.5	2	6	4	1	
a_{37}	N	100.5	7	6	4	2	
a_{38}	Y	59.9	7	6	1	2	2
a_{39}	Y	99.3	5	1	1	1	4
a_{40}	Y	24.7	3	4	1	1	4
a_{41}	N	33.0	3	4	4	1	
a_{42}	N	117.6	5	4	4	1	
a_{43}	N	207.0	7	6	1	2	
a_{44}	N	431.3	5	4	3	1	
a_{45}	Y	62.0	3	6	3	1	3
a_{46}	Y	283.9	5	2	1	1	3
a_{48}	Y	54.1	2	6	3	1	3
a_{50}	N	21.8	2	6	4	1	
a_{52}	N	6.7	1	4	1	1	
a_{53}	N	31.3	2	1	4	1	
a_{54}	Y	98.6	1	1	3	1	4
a_{55}	Y	60.4	2	6	4	1	3
a_{56}	N	112.7	2	6	4	1	
a_{59}	Y	576.7	7	6	1	5	1
a_{75}	N	273.9	7	6	1	1	
a_{76}	N	289.6	7	6	1	8	
a_{83}	N	84.5	1	6	6	1	
a_{85}	N	264.8	1	6	4	1	
a_{86}	N	120.8	1	6	4	1	
a_{87}	N	103.7	1	6	6	1	
a_{88}	N	33.8	1	1	1	1	
a_{89}	N	54.3	5	2	4	1	
a_{90}	N	75.9	3	1	1	1	
a_{92}	N	12.0	7	6	1	1	
a_{93}	N	8.0	5	6	1	1	
a_{94}	N	51.6	7	6	1	1	
a_{95}	N	4.4	3	5	4	1	
a_{96}	N	38.9	3	6	1	1	
a_{97}	N	34.8	1	4	3	1	
a_{99}	N	50.8	3	4	3	1	
a_{100}	N	104.0	3	4	4	1	
a_{106}	N	91.4	7	6	1	2	
a_{107}	N	135.8	5	6	6	1	
a_{108}	Y	108.2	5	6	4	1	2
a_{109}	Y	213.4	5	6	3	1	2
a_{111}	Y	95.9	5	6	1	1	3
a_{112}	N	466.6	7	6	1	1	

$$\underline{P}(Cl_h^{\geqslant}) = \{a \in A^R : L^{DM}(D_P^+(a)) \geqslant h\}; \quad (4)$$
$$\overline{P}(Cl_h^{\geqslant}) = \{a \in A^R : R^{DM}(D_P^+(a)) \geqslant h\}, \quad (5)$$

where $L^{DM}(D_P^+(a)) = min\{L^{DM}(b) : b \in D_P^+(a)\}$ and $R^{DM}(D_P^+(a)) = max\{R^{DM}(b) : b \in D_P^+(a)\}$. Analogously, P-lower and P-upper approximations of $Cl_h^{\leqslant},\ h = 1, \ldots, p-1$, with respect to $P \subseteq G$, are defined, respectively, as:

$$\underline{P}(Cl_h^{\leqslant}) = \{a \in A^R : R^{DM}(D_P^-(a)) \leqslant h\}; \quad (6)$$
$$\overline{P}(Cl_h^{\leqslant}) = \{a \in A^R : L^{DM}(D_P^-(a)) \leqslant h\}, \quad (7)$$

where $L^{DM}(D_P^-(a)) = max\{R^{DM}(b) : b \in D_P^-(a)\}$ and $R^{DM}(D_P^-(a)) = min\{L^{DM}(b) : b \in D_P^-(a)\}$. Finally, the P-boundaries of $Cl_h^{\geqslant}$ and $Cl_h^{\leqslant}$ are defined as:

$$Bn(Cl_h^{\geqslant}) = \overline{P}(Cl_h^{\geqslant}) \setminus \underline{P}(Cl_h^{\geqslant}); \quad (8)$$
$$Bn(Cl_h^{\leqslant}) = \overline{P}(Cl_h^{\leqslant}) \setminus \underline{P}(Cl_h^{\leqslant}). \quad (9)$$

llustrative example (part 2): rough approximations of class unions

The provided 23 assignment examples are consistent, i.e. there is no dominated alternative assigned by the DM to a class better than the alternative that dominates it. The lower approximations of class unions are presented in Table 2. Obviously, in this case they are equal to upper approximations.

3. Generating all compatible minimal rules

The decision rules are expressions of the form:

if *condition*, **then** *consequent*

that represent a form of dependency between condition criteria and the decision concerning assignment of the alternative. We say that an alternative supports a decision rule if it matches both condition and decision parts of the rule. On the other hand, an alternative is covered by a decision rule if it matches the condition part of the rule. We can distinguish three types of decision rules: certain, possible and approximate. The certain and possible rules are generated, respectively, from the lower and upper approximations of class unions, whereas the approximate decision rules are generated from the boundary regions. In the following, when different is not explicitly stated, we will consider only certain rules.

Each decision rule indicates a class union (either with the lower or the upper limit) for a covered alternative $a \in A \setminus A^R$. In order to induce certain decision rules with consequent $a \in Cl_h^{\geqslant}$ or $a \in Cl_h^{\leqslant}$, one needs to consider positive examples, i.e. reference alternatives concordant with these conclusions (alternatives from the lower approximations $\underline{P}(Cl_h^{\geqslant})$ and $\underline{P}(Cl_h^{\leqslant})$, respectively), and negative examples (reference alternatives not belonging to these lower approximations, i.e., respectively $\overline{P}(Cl_h^{\leqslant})$ and $\overline{P}(Cl_h^{\geqslant})$). Considering upward and downward unions we can distinguish two types of rules:

- $D_{\geqslant}$-decision rules with the following syntax:

$$\text{if } g_1(a) \geqslant r_{g_1} \text{ and } g_2(a) \geqslant r_{g_2} \text{ and } \ldots \text{ and } g_p(a) \geqslant r_{g_p}, \quad \text{then } a \in Cl_h^{\geqslant}, \quad (10)$$

- $D_{\leqslant}$-decision rules with the following syntax:

$$\text{if } g_1(a) \leqslant r_{g_1} \text{ and } g_2(a) \leqslant r_{g_2} \text{ and } \ldots \text{ and } g_p(a) \leqslant r_{g_p}, \quad \text{then } a \in Cl_h^{\leqslant}, \quad (11)$$

where $P = \{g_1, g_2, \ldots, g_p\} \subseteq G, (r_{q_1}, \ldots, r_{q_p}) \in X_1 \times \ldots \times X_p$ and $h \in H$.

Table 2
Lower approximations of upward and downward class unions.

$\underline{P}(C_2^{\geqslant})$	$\{a_6, a_{27}, a_{38}, a_{108}, a_{109}, a_{12}, a_{30}, a_{32}, a_{33}, a_{45}, a_{46}, a_{48}, a_{55}, a_{111}, a_1, a_3, a_{15}, a_{31}, a_{39}, a_{40}, a_{54}\}$
$\underline{P}(C_3^{\geqslant})$	$\{a_{12}, a_{30}, a_{32}, a_{33}, a_{45}, a_{46}, a_{48}, a_{55}, a_{111}, a_1, a_3, a_{15}, a_{31}, a_{39}, a_{40}, a_{54}\}$
$\underline{P}(C_4^{\geqslant})$	$\{a_1, a_3, a_{15}, a_{31}, a_{39}, a_{40}, a_{54}\}$
$\underline{P}(C_1^{\leqslant})$	$\{a_7, a_{59}\}$
$\underline{P}(C_2^{\leqslant})$	$\{a_6, a_{27}, a_{38}, a_{108}, a_{109}, a_7, a_{59}\}$
$\underline{P}(C_3^{\leqslant})$	$\{a_{12}, a_{30}, a_{32}, a_{33}, a_{45}, a_{46}, a_{48}, a_{55}, a_{111}, a_6, a_{27}, a_{38}, a_{108}, a_{109}, a_7, a_{59}\}$

Each decision rule should be minimal, i.e., its condition should be composed of a minimal number of the weakest elementary conditions referring to values on particular criteria and the strongest consequent.

Definition 3.1. Decision rule is **minimal** if there is no other rule with an antecedent of at least the same weakness (i.e., a rule using a subset of elementary conditions or/and weaker elementary conditions) and a consequent of at least the same strength (i.e., a rule assigning objects to the same union or sub-union of classes).

Consider a $D_{\geqslant}$-decision rule "if $g_1(a) \geqslant r_{g_1}$ and $g_2(a) \geqslant r_{g_2}$ and $\ldots g_p(a) \geqslant r_{g_p}$, then $a \in Cl_t^{\geqslant}$". If there exists an alternative $y \in \underline{P}(Cl_t^{\geqslant})$ such that $g_1(a) = r_{g_1}$ and $g_2(a) = r_{g_2}$ and $\ldots g_p(a) = r_{g_p}$, then y is called basis of the rule. Each $D_{\geqslant}$-decision rule having a basis is called robust because it is "built" on an existing reference alternative. Although algorithms which construct robust rules reduce the computational complexity, they usually generate a greater number of rules instead of few more general ones. Thus, in this paper, we rather focus on "mix of conditions" rules, which are possibly founded by multiple reference alternatives.

In the following, we discuss an algorithm which generates all certain decision rules for the lower approximation of upward union of classes $\underline{P}(Cl_h^{\geqslant}), h = 2, \ldots, p$. The algorithm for the downward union of classes $\underline{P}(Cl_h^{\leqslant}), h = 1, \ldots, p-1$, can be formulated analogously. In the first phase (see Algorithm 1), we generate the elementary conditions to be used in the construction of decision rules. We take advantage of the following proposition.

Proposition 3.1. *If there exist a minimal certain rule r_1 in form:*

$$\text{if } g_1(a) \geqslant r_{g_1} \text{ and } g_2(a) \geqslant r_{g_2} \text{ and } \ldots \text{ and } g_p(a) \geqslant r_{g_p}, \quad \text{then } a \in Cl_h^{\geqslant},$$

and another rule r_2 in form:

$$\text{if } g_1(a) \geqslant r^*_{g_1} \text{ and } g_2(a) \geqslant r^*_{g_2} \text{ and } \ldots \text{ and } g_p(a) \geqslant r^*_{g_p}, \quad \text{then } a \in Cl_h^{\geqslant},$$

where $r^*_{g_j} \leqslant r_{g_j}$, and negative supports for each pair of elementary conditions $(g_j(a) \geqslant r^*_{g_j})$ and $(g_j(a) \geqslant r_{g_j}), j = 1, \ldots, p$, are equal, then rule r_2 covers all positive examples covered by rule r_1 and does not cover any negative example.

Algorithm 1. Generation of the elementary conditions to be used in the decision rules concerning the lower approximation of upward union of classes $\underline{P}(Cl_h^{\geqslant})$.

Input $\underline{P}(Cl_h^{\geqslant})$: the lower approximation of upward union of classes $Cl_h^{\geqslant}$
Output C_1: a set of elementary conditions to be used in the decision rules concerning $\underline{P}(Cl_h^{\geqslant})$
$C_1 = \emptyset$.
for $g_j \in G$
  for $a_i^* \in \underline{P}(Cl_h^{\geqslant})$ **do**
    $(g_j(a) \geqslant g_j(a_i^*))$.positive-support = $|\, a_k^* \in \underline{P}(Cl_h^{\geqslant}) : g_j(a_k^*) \geqslant g_j(a_i^*) \,|$.
    $(g_j(a) \geqslant g_j(a_i^*))$.negative-support = $|\, a_k^* \in A^R \setminus \underline{P}(Cl_h^{\geqslant}) : g_j(a_k^*) \geqslant g_j(a_i^*) \,|$.
    $(g_j(a) \geqslant g_j(a_i^*))$.criterion = j.
  end for
end for
for $g_j \in G$ **do**
  for $a_i^* \in \underline{P}(Cl_h^{\geqslant})$ **do**
    if $(g_j(a) \geqslant g_j(a_i^*)) \notin C_1$ and $\nexists a_k^* \in \underline{P}(Cl_h^{\geqslant}), k \neq i : (g_j(a_k^*) < g_j(a_i^*))$ and $(g_j(a) \geqslant g_j(a_k^*))$.
    negative-support $= (g_j(a) \geqslant g_j(a_i^*))$.negative-support **then**
      $C_1 = C_1 \cup (g_j(a) \geqslant g_j(a_i^*))$.
    end if
  end for
end for

Thus, to reduce the number of conditions that are subsequently combined into conjunctions, we consider only conditions $(g_j(a) \geqslant g_j(a_i^*))$, where $a_i^* \in \underline{P}(Cl_h^{\geqslant})$ and $g_j(a_i^*)$ is minimal in the set of different performances whose corresponding elementary conditions have the same negative support. With each elementary condition $(g_j(a) \geqslant g_j(a_i^*))$, we associate three parameters:

- $(g_j(a) \geqslant g_j(a_i^*))$.positive-support – a number of reference alternatives in $\underline{P}(Cl_h^{\geqslant})$ (i.e. positive examples) covered by the condition;
- $(g_j(a) \geqslant g_j(a_i^*))$.negative-support – a number of reference alternatives in $A^R \setminus \underline{P}(Cl_h^{\geqslant})$ (i.e. negative examples) covered by the condition;
- $(g_j(a) \geqslant g_j(a_i^*))$.criterion – an index of the corresponding criterion.

Algorithm 2. Generation of a set of conjunctions of k elementary conditions to be used in the decision rules concerning $\underline{P}(Cl_h^{\geqslant})$.

Input $\underline{P}(Cl_h^{\geqslant})$: the lower approximation of upward union of classes $Cl_h^{\geqslant}$
Input C_{k-1}: a set of conjunctions c_{k-1} of $k-1$ elementary conditions covering at least one reference alternative in $\underline{P}(Cl_h^{\geqslant})$
Output C_k: a set of conjunctions c_k of k elementary conditions covering at least one reference alternative in $\underline{P}(Cl_h^{\geqslant})$

for $c_{k-1}^i \in C_{k-1}$ **do**
  for $c_{k-1}^j \in C_{k-1}$ **do**
    if c_{k-1}^i.negative-support > 0 **and** c_{k-1}^j.negative-support > 0 **and** $c_{k-1}^i.condition_l = c_{k-1}^j.condition_l, l = 1, \ldots, k-2$ **and** $c_{k-1}^i.condition_{k-1} < c_{k-1}^j.condition_{k-1}$ **and** $c_{k-1}^i.condition_{k-1}, c_{k-1}^j.condition_{k-1}$ do not concern the same criterion **then**
      $c_k.conditions = (c_{k-1}^i.condition_l, l = 1, \ldots, k-2,\ c_{k-1}^i.condition_{k-1},\ c_{k-1}^j.condition_{k-1})$.
      c_k.positive-support = $|\ a_i^* \in \underline{P}(Cl_h^{\geqslant}) : g_{c_k.condition_j.criterion}(a_i^*)$satisfies$c_k.condition_j$, for $j = 1, \ldots, k\ |$.
      c_k.negative-support = $|\ a_i^* \in A^R \setminus \underline{P}(Cl_h^{\geqslant}) : g_{c_k.condition_j.criterion}(a_i^*)$satisfies$c_k.condition_j$, for $j = 1, \ldots, k\ |$.
      if c_k.positive-support $\geqslant 1$ **then**
        $C_k = C_k \cup c_k$.
      end if
    end if
  end for
end for

Then, we generate a set of conjunctions of elementary conditions which cover at least one reference alternative in $\underline{P}(Cl_h^{\geqslant})$ (see Algorithm 2). Analogously to the DOMApriori algorithm [42], we assume that elementary conditions are ordered lexicographically, that in the constructed conjunctions the indices of the conditions used are ordered increasingly, and that no criterion can be used in the conjunction more than once. With each conjunction c_k of size k, we associate the following parameters:

- $c_k.conditions$, where *conditions* represents the set of k elementary conditions $\{condition_1, \ldots, condition_k\}$ used in the conjunction c_k; each condition $condition_l,\ l = 1, \ldots, k$, has the following form: $(g_j(a) \geqslant g_j(a_i^*)),\ j \in j, a_i^* \in A^R$, and is described with three aforementioned condition-specific parameters: "positive-support", "negative-support", and "criterion"; when referring to the l-th elementary condition $(l = 1, \ldots, k)$ in c_k, we will use the following notation: $c_k.condition_l$;
- c_k.positive-support – a number of reference alternatives in $\underline{P}(Cl_h^{\geqslant})$ covered by all elementary conditions in c_k;
- c_k.negative-support – a number of reference alternatives in $A^R \setminus \underline{P}(Cl_h^{\geqslant})$ covered by all elementary conditions in c_k.

Each conjunction of size k is obtained by merging a pair of conjunctions of size $k-1$ which contain the same $k-2$ conditions, thus, differing by just a single one. These differentiating conditions need to concern different criteria. Conjunctions of size $k-1$ with negative support equal to 0 are not taken into account when building conjunctions of size k, since they already contain all conditions necessary for discriminating reference alternatives in $\underline{P}(Cl_h^{\geqslant})$ and $A^R \setminus \underline{P}(Cl_h^{\geqslant})$. Finally, the set C_k of conjunctions of size k contains only these conjunctions whose positive support is greater than 0.

Algorithm 3. Generation of a decision rule concerning $\underline{P}(Cl_h^{\geqslant})$ on the basis of the provided conjunction of elementary conditions.

Input $\underline{P}(Cl_h^{\geqslant})$: the lower approximation of upward union of class $Cl_h^{\geqslant}$
Input c_k: conjunction of k elementary conditions
Output r_{c_k}: minimal decision rule covering at least one reference alternative in $\underline{P}(Cl_h^{\geqslant})$

for $j = 1, \ldots, k$ **do**
  x_l = minimal evaluation on $g_{(c_k.condition_j.criterion)}$ of any $a^* \in \underline{P}(Cl_h^{\geqslant})$ that satisfies all conditions in c_k.
  $r_{c_k}.condition_j$ = $(g_{(c_k.condition_j.criterion)}(a) \geqslant x_l)$.
end for
r_{c_k}.decision-part = $(a \in Cl_h^{\geqslant})$.

The procedure for generating all minimal decision rules concerning $\underline{P}(Cl_h^{\geqslant})$ is summarized as Algorithm 4. After generating all possible conjunctions of elementary conditions covering at least one reference alternative in $\underline{P}(Cl_h^{\geqslant})$, we need to eliminate conjunctions covering any negative example in $A^R \setminus \underline{P}(Cl_h^{\geqslant})$. Subsequently, we remove conjunctions of conditions which are not minimal, i.e. such that there exists some other conjunction using a subset of elementary conditions or/and weaker elementary conditions. Since conjunctions are composed of conditions $(g_j(a) \geqslant g_j(a_i^*))$, such that $g_j(a_i^*)$ is minimal in the set of different evaluations in $\underline{P}(Cl_h^{\geqslant})$ whose corresponding elementary conditions have the same negative support, at the end of the procedure each rule is constructed so as to be built on reference alternatives satisfying the conjunction of conditions to be used in the condition part (i.e., so that the elementary conditions refer to the evaluations of the reference alternatives supporting the rule). This idea is formalized in Algorithm 3. It generates a single rule r_{c_k} from the conjunction of elementary conditions c_k. Each condition used in the condition part of r_{c_k}, denoted by $r_{c_k}.condition_j$, $j = 1, \ldots, k$, is derived from the above described transformation of $c_k.condition_j$. The decision part of r_{c_k}, denoted by r_{c_k}.decision-part, is $(a \in Cl_h^{\geqslant})$.

Algorithm 4. Generation of all minimal decision rules concerning alternatives from $\underline{P}(Cl_h^{\geqslant})$.

Input $\underline{P}(Cl_h^{\geqslant})$: the lower approximation of upward union of class $Cl_h^{\geqslant}$
Output C_k: conjunctions of k conditions with positive support in $\underline{P}(Cl_h^{\geqslant})$
Execute Algorithm 1 for $\underline{P}(Cl_h^{\geqslant})$ to obtain C_1.
$k = 1$.
while $C_k \neq \emptyset$ **do**
  Execute Algorithm 2 for $\underline{P}(Cl_h^{\geqslant})$ and C_k to obtain C_{k+1}.
  $k = k + 1$.
end while
max = k.
for $j = 1, \ldots, max$ **do**
  for $c_j \in C_j$ **do**
    if c_j.negative-support $\geqslant 1$ **then**
      $C_j = C_j \setminus c_j$.
    end if
  end for
end for
for $j = 1, \ldots, max$ **do**
  for $c_j \in C_j$ **do**
    if c_j is not a minimal set of conditions **then**
      $C_j = C_j \setminus c_j$.
    end if
  end for
end for
for $j = 1, \ldots, max$ **do**
  for $c_j \in C_j$ **do**
    Execute Algorithm 3 for $\underline{P}(Cl_h^{\geqslant})$ and c_j to obtain rule r_{c_j}.
    $\mathcal{R}_{all}^{\underline{P}(Cl_h^{\geqslant})} = \mathcal{R}_{all}^{\underline{P}(Cl_h^{\geqslant})} \cup r_{c_j}$.
  end for
end for

Although our aim is to generate all rules compatible with the preference information provided by the DM, the above algorithm can be easily adjusted to account for the required minimal support and maximal length (i.e., number of elementary conditions) of the induced rules.

Illustrative example (part 3): all compatible minimal rules

Using Algorithm 4 for inducing all minimal rules $\mathcal{R}_{all}^{A^R}$ compatible with the lower approximations presented in Table 2 leads to a set of 44 certain rules (25 and 19 rules for the upward and downward class unions, respectively). These are listed in Table 3. For example, there are 3 rules with a decision part $(a \in Cl_1^{\leqslant})$ and 12 rules with a decision part $(a \in Cl_4^{\geqslant})$. When it comes to the number of elementary conditions, there are 17 rules with just a single condition, 19 rules with two ones, and 8 rules with three conditions. Among these elementary conditions there are $22, 19, 13, 17$, and 13 ones which refer to criterion g_1, g_2, g_3, g_4, and g_5, respectively.

4. Generating all compatible minimal-cover (MC) sets of rules

Definition 4.1. A set of certain decision rules is **complete** if the following conditions are fulfilled: each $y \in \underline{P}(Cl_t^{\geqslant})$ supports at least one certain $D_{\geqslant}$-decision rule whose consequent is "$a \in Cl_r^{\geqslant}$", with $r, t \in \{2, \ldots, p\}$ and $r \geqslant t$ and each $y \in \underline{P}(Cl_t^{\leqslant})$ supports at least one certain $D_{\leqslant}$-decision rule whose consequent is "$a \in Cl_u^{\leqslant}$", with $u, t \in \{1, \ldots, p-1\}$ and $u \leqslant t$.

Thus, complete means that the set of rules is able to cover all reference alternatives, so that reference alternatives with precise assignments are reassigned to the specified classes and reference alternatives with imprecise assignments are assigned to a set of contiguous classes indicated by the DM.

Table 3
All compatible certain minimal rules.

Symbol	Rule
$r^1_{\geqslant 2}$	if $(g_3 \leqslant 4)$ then $a \in Cl_2^{\leqslant}$
$r^2_{\geqslant 2}$	if $(g_5 \leqslant 2)$ then $a \in Cl_2^{\leqslant}$
$r^3_{\geqslant 2}$	if $(g_1 \leqslant 163.1)$ then $a \in Cl_2^{\leqslant}$
$r^4_{\geqslant 2}$	if $(g_2 \leqslant 5)$ then $a \in Cl_2^{\leqslant}$
$r^1_{\geqslant 3}$	if $(g_3 \leqslant 4)$ then $a \in Cl_3^{\leqslant}$
$r^2_{\geqslant 3}$	if $(g_1 \leqslant 49.8)$ then $a \in Cl_3^{\leqslant}$
$r^3_{\geqslant 3}$	if $(g_2 \leqslant 3)$ then $a \in Cl_3^{\leqslant}$
$r^4_{\geqslant 3}$	if $(g_1 \leqslant 54.1)$ and $(g_4 \leqslant 3)$ then $a \in Cl_3^{\leqslant}$
$r^5_{\geqslant 3}$	if $(g_1 \leqslant 99.3)$ and $(g_5 \leqslant 1)$ then $a \in Cl_3^{\leqslant}$
$r^6_{\geqslant 3}$	if $(g_2 \leqslant 5)$ and $(g_4 \leqslant 1)$ then $a \in Cl_3^{\leqslant}$
$r^7_{\geqslant 3}$	if $(g_1 \leqslant 99.3)$ and $(g_2 \leqslant 5)$ then $a \in Cl_3^{\leqslant}$
$r^8_{\geqslant 3}$	if $(g_1 \leqslant 150.4)$ and $(g_4 \leqslant 3)$ and $(g_5 \leqslant 1)$ then $a \in Cl_3^{\leqslant}$
$r^9_{\geqslant 3}$	if $(g_1 \leqslant 150.4)$ and $(g_2 \leqslant 5)$ and $(g_4 \leqslant 3)$ then $a \in Cl_3^{\leqslant}$
$r^1_{\geqslant 4}$	if $(g_1 \leqslant 24.7)$ then $a \in Cl_4^{\leqslant}$
$r^2_{\geqslant 4}$	if $(g_2 \leqslant 2)$ and $(g_3 \leqslant 1)$ then $a \in Cl_4^{\leqslant}$
$r^3_{\geqslant 4}$	if $(g_2 \leqslant 1)$ and $(g_4 \leqslant 3)$ then $a \in Cl_4^{\leqslant}$
$r^4_{\geqslant 4}$	if $(g_3 \leqslant 1)$ and $(g_4 \leqslant 3)$ then $a \in Cl_4^{\leqslant}$
$r^5_{\geqslant 4}$	if $(g_2 \leqslant 3)$ and $(g_4 \leqslant 1)$ then $a \in Cl_4^{\leqslant}$
$r^6_{\geqslant 4}$	if $(g_1 \leqslant 66.4)$ and $(g_2 \leqslant 1)$ then $a \in Cl_4^{\leqslant}$
$r^7_{\geqslant 4}$	if $(g_1 \leqslant 99.3)$ and $(g_3 \leqslant 1)$ then $a \in Cl_4^{\leqslant}$
$r^8_{\geqslant 4}$	if $(g_1 \leqslant 66.4)$ and $(g_2 \leqslant 2)$ and $(g_3 \leqslant 4)$ then $a \in Cl_4^{\leqslant}$
$r^9_{\geqslant 4}$	if $(g_2 \leqslant 3)$ and $(g_3 \leqslant 4)$ and $(g_4 \leqslant 3)$ then $a \in Cl_4^{\leqslant}$
$r^{10}_{\geqslant 4}$	if $(g_1 \leqslant 99.3)$ and $(g_3 \leqslant 4)$ and $(g_4 \leqslant 3)$ then $a \in Cl_4^{\leqslant}$
$r^{11}_{\geqslant 4}$	if $(g_1 \leqslant 66.4)$ and $(g_4 \leqslant 1)$ and $(g_5 \leqslant 1)$ then $a \in Cl_4^{\leqslant}$
$r^{12}_{\geqslant 4}$	if $(g_1 \leqslant 66.4)$ and $(g_2 \leqslant 5)$ and $(g_4 \leqslant 1)$ then $a \in Cl_4^{\leqslant}$
$r^1_{\leqslant 1}$	if $(g_1 \geqslant 576.7)$ then $a \in Cl_1^{\leqslant}$
$r^2_{\leqslant 1}$	if $(g_5 \geqslant 5)$ then $a \in Cl_1^{\leqslant}$
$r^3_{\leqslant 1}$	if $(g_1 \geqslant 208.9)$ and $(g_2 \geqslant 7)$ then $a \in Cl_1^{\leqslant}$
$r^1_{\leqslant 2}$	if $(g_2 \geqslant 7)$ then $a \in Cl_2^{\leqslant}$
$r^2_{\leqslant 2}$	if $(g_5 \geqslant 2)$ then $a \in Cl_2^{\leqslant}$
$r^3_{\leqslant 2}$	if $(g_1 \geqslant 576.7)$ then $a \in Cl_2^{\leqslant}$
$r^4_{\leqslant 2}$	if $(g_1 \geqslant 108.2)$ and $(g_3 \geqslant 6)$ then $a \in Cl_2^{\leqslant}$
$r^5_{\leqslant 2}$	if $(g_1 \geqslant 163.1)$ and $(g_4 \geqslant 3)$ then $a \in Cl_2^{\leqslant}$
$r^6_{\leqslant 2}$	if $(g_2 \geqslant 5)$ and $(g_4 \geqslant 3)$ then $a \in Cl_2^{\leqslant}$
$r^1_{\leqslant 3}$	if $(g_2 \geqslant 7)$ then $a \in Cl_3^{\leqslant}$
$r^2_{\leqslant 3}$	if $(g_4 \geqslant 4)$ then $a \in Cl_3^{\leqslant}$
$r^3_{\leqslant 3}$	if $(g_5 \geqslant 2)$ then $a \in Cl_3^{\leqslant}$
$r^4_{\leqslant 3}$	if $(g_1 \geqslant 108.2)$ then $a \in Cl_3^{\leqslant}$
$r^5_{\leqslant 3}$	if $(g_2 \geqslant 3)$ and $(g_3 \geqslant 6)$ then $a \in Cl_3^{\leqslant}$
$r^6_{\leqslant 3}$	if $(g_3 \geqslant 6)$ and $(g_4 \geqslant 3)$ then $a \in Cl_3^{\leqslant}$
$r^7_{\leqslant 3}$	if $(g_1 \geqslant 49.8)$ and $(g_3 \geqslant 2)$ then $a \in Cl_3^{\leqslant}$
$r^8_{\leqslant 3}$	if $(g_2 \geqslant 3)$ and $(g_4 \geqslant 3)$ then $a \in Cl_3^{\leqslant}$
$r^9_{\leqslant 3}$	if $(g_2 \geqslant 5)$ and $(g_3 \geqslant 2)$ then $a \in Cl_3^{\leqslant}$
$r^{10}_{\leqslant 3}$	if $(g_1 \geqslant 49.8)$ and $(g_2 \geqslant 2)$ and $(g_4 \geqslant 3)$ then $a \in Cl_3^{\leqslant}$

Definition 4.2. A set of certain decision rules is **minimal-cover** (MC) if and only if it is complete and non-redundant, i.e., exclusion of any rule from this set makes it non-complete.

Finding a minimum set of rules covering the lower approximation of upward (downward) union of classes $\underline{P}(Cl_h^{\geqslant})$ ($\underline{P}(Cl_h^{\leqslant})$) is analogous to solving the minimum set cover problem [48]. This classical problem in combinatorics and computer science is defined in the following way.

Definition 4.3 (*Minimum set cover problem*). Given a set of elements $\mathcal{U} = \{1, 2, \ldots, m\}$ (called the universe) and a collection of subsets of $\mathcal{U}, U_1, \ldots, U_k$, whose union comprises the universe $\mathcal{U}$, the minimum set cover problem is to identify the minimum set of subsets $U_i, 1, \ldots, k$, whose union is $\mathcal{U}$.

The study of the set cover problem has led to the development of fundamental techniques for the field of approximation algorithms. In fact, many combinatorial problems can be viewed as either special cases or dual problems of minimal set cover. Assuming $\mathcal{U} = \underline{P}(Cl_h^{\geqslant})$ ($\mathcal{U} = \underline{P}(Cl_h^{\leqslant})$) and $U_i = cov(r_i)$ for $r_i \in \mathcal{R}_{all}^{\underline{P}(Cl_h^{\geqslant})}$ ($r_i \in \mathcal{R}_{all}^{\underline{P}(Cl_h^{\leqslant})}$) and $i = 1, \ldots, k$, the problem of selecting a minimum set of rules covering the lower approximation of upward (downward) union of classes $\underline{P}(Cl_h^{\geqslant})$ ($\underline{P}(Cl_h^{\leqslant})$) can be formulated in the following way.

Definition 4.4 (*Minimum-cover set of rules for the lower approximation of upward (downward) union of classes* $\underline{P}(Cl_h^{\geqslant})$ ($\underline{P}(Cl_h^{\leqslant})$)). Given the lower approximation of upward (downward) union of classes $\underline{P}(Cl_h^{\geqslant})$ ($\underline{P}(Cl_h^{\leqslant})$) and a collection of all minimal rules $\mathcal{R}_{all}^{\underline{P}(Cl_h^{\geqslant})} = \{r_1, r_2, \ldots, r_k\}$ ($\mathcal{R}_{all}^{\underline{P}(Cl_h^{\leqslant})}$) covering reference alternatives in $\underline{P}(Cl_h^{\geqslant})$ ($\underline{P}(Cl_h^{\leqslant})$), the problem of finding minimum-cover set of rules amounts at identifying the minimum subset of rules in $\mathcal{R}_{all}^{\underline{P}(Cl_h^{\geqslant})}$ ($\mathcal{R}_{all}^{\underline{P}(Cl_h^{\leqslant})}$) which cover $\underline{P}(Cl_h^{\geqslant})$ ($\underline{P}(Cl_h^{\leqslant})$).

All minimal-cover sets of rules for the lower approximation of upward union of classes $\underline{P}(Cl_h^{\geqslant})$ can be identified using Algorithm 5. It formulates and solves a series of Integer Linear Programming (ILP) problems which guarantee that each reference alternative $a_i^* \in A^R$ is covered by at least one rule from $\mathcal{R}_{all}^{\underline{P}(Cl_h^{\geqslant})}$ (see Step 8). The MC sets of rules in the subsequent iterations ($k > 1$) are discovered by adding constraints that forbid finding again already identified solutions (see Step 15). Algorithm 5 should be executed for $\underline{P}(Cl_h^{\geqslant}),\ h = 2, \ldots, p$, while its suitably adapted version for downward union of classes needs to be executed for $\underline{P}(Cl_h^{\leqslant}),\ h = 1, \ldots, p-1$. Obviously, this algorithm can be easily adapted to generate only sets of rules with an upper threshold t_{up} on the number of rules by requiring that the sum of all binary variables v_i associated with the rules is not greater than t_{up}.

Finally, all compatible minimal sets of rules $\mathcal{R}^{A^R}$ are formed by the following product:

$$\mathcal{R}^{A^R} = \mathcal{R}_{mrc}^{\underline{P}(Cl_2^{\geqslant})} \times \cdots \times \mathcal{R}_{mrc}^{\underline{P}(Cl_p^{\geqslant})} \times \mathcal{R}_{mrc}^{\underline{P}(Cl_1^{\leqslant})} \times \cdots \times \mathcal{R}_{mrc}^{\underline{P}(Cl_{p-1}^{\leqslant})}. \tag{12}$$

When computing each MC rule set in $\mathcal{R}^{A^R}$ according to (12), we should eliminate decision rules from $\mathcal{R}_{mrc}^{\underline{P}(Cl_h^{\geqslant})}$ or $\mathcal{R}_{mrc}^{\underline{P}(Cl_h^{\leqslant})}$ with a consequent having at least the same strength (i.e., rules assigning objects to the same union or sub-union of classes) as some other rules from, respectively $\mathcal{R}_{mrc}^{\underline{P}(Cl_t^{\geqslant})}, t > h$, or $\mathcal{R}_{mrc}^{\underline{P}(Cl_t^{\leqslant})}, t < h$. This step can be skipped if the DM is interested only in the analysis of the sorting results, because it does not influence outcome of the sorting method (see Section 5).

Table 4
All minimal-cover sets of rules for the lower approximations of class unions.

	Cardinality	Minimal-cover sets of rules
$\mathcal{R}_{mrc}^{\underline{P}(Cl_1^{\leqslant})}$	2	$\{r^2_{\leqslant 1}\}, \{r^3_{\leqslant 1}\}$
$\mathcal{R}_{mrc}^{\underline{P}(Cl_2^{\leqslant})}$	3	$\{r^2_{\leqslant 2}, r^4_{\leqslant 2}\}, \{r^1_{\leqslant 2}, r^5_{\leqslant 2}\}, \{r^1_{\leqslant 2}, r^4_{\leqslant 2}\}$
$\mathcal{R}_{mrc}^{\underline{P}(Cl_3^{\leqslant})}$	8	$\{r^7_{\leqslant 3}, r^{10}_{\leqslant 3}\}, \{r^2_{\leqslant 3}, r^7_{\leqslant 3}\}, \{r^4_{\leqslant 3}, r^7_{\leqslant 3}\}, \{r^7_{\leqslant 3}, r^8_{\leqslant 3}\}, \{r^2_{\leqslant 3}, r^9_{\leqslant 3}, r^{10}_{\leqslant 3}\}, \{r^2_{\leqslant 3}, r^6_{\leqslant 3}, r^9_{\leqslant 3}\},$
		$\{r^2_{\leqslant 3}, r^4_{\leqslant 3}, r^5_{\leqslant 3}, r^{10}_{\leqslant 3}\}, \{r^2_{\leqslant 3}, r^4_{\leqslant 3}, r^5_{\leqslant 3}, r^6_{\leqslant 3}\}$
$\mathcal{R}_{mrc}^{\underline{P}(Cl_2^{\geqslant})}$	2	$\{r^2_{\geqslant 2}\}, \{r^3_{\geqslant 2}, r^4_{\geqslant 2}\}$
$\mathcal{R}_{mrc}^{\underline{P}(Cl_3^{\geqslant})}$	5	$\{r^1_{\geqslant 3}, r^7_{\geqslant 3}\}, \{r^3_{\geqslant 3}, r^6_{\geqslant 3}\}, \{r^1_{\geqslant 3}, r^5_{\geqslant 3}\}, \{r^1_{\geqslant 3}, r^3_{\geqslant 3}, r^8_{\geqslant 3}\}, \{r^1_{\geqslant 3}, r^3_{\geqslant 3}, r^9_{\geqslant 3}\}$
$\mathcal{R}_{mrc}^{\underline{P}(Cl_4^{\geqslant})}$	18	$\{r^1_{\geqslant 4}, r^7_{\geqslant 4}\}, \{r^5_{\geqslant 4}, r^{10}_{\geqslant 4}\}, \{r^1_{\geqslant 4}, r^{10}_{\geqslant 4}\}, \{r^{10}_{\geqslant 4}, r^{11}_{\geqslant 4}\}, \{r^1_{\geqslant 4}, r^4_{\geqslant 4}\}, \{r^{10}_{\geqslant 4}, r^{12}_{\geqslant 4}\}, \{r^5_{\geqslant 4}, r^7_{\geqslant 4}, r^9_{\geqslant 4}\},$
		$\{r^5_{\geqslant 4}, r^7_{\geqslant 4}, r^8_{\geqslant 4}\}, \{r^4_{\geqslant 4}, r^5_{\geqslant 4}, r^9_{\geqslant 4}\}, \{r^4_{\geqslant 4}, r^5_{\geqslant 4}, r^8_{\geqslant 4}\}, \{r^7_{\geqslant 4}, r^9_{\geqslant 4}, r^{11}_{\geqslant 4}\}, \{r^7_{\geqslant 4}, r^9_{\geqslant 4}, r^{12}_{\geqslant 4}\}, \{r^7_{\geqslant 4}, r^8_{\geqslant 4}, r^{11}_{\geqslant 4}\},$
		$\{r^7_{\geqslant 4}, r^8_{\geqslant 4}, r^{12}_{\geqslant 4}\}, \{r^4_{\geqslant 4}, r^9_{\geqslant 4}, r^{12}_{\geqslant 4}\}, \{r^4_{\geqslant 4}, r^9_{\geqslant 4}, r^{11}_{\geqslant 4}\}, \{r^4_{\geqslant 4}, r^8_{\geqslant 4}, r^{11}_{\geqslant 4}\}, \{r^4_{\geqslant 4}, r^8_{\geqslant 4}, r^{12}_{\geqslant 4}\}$

Algorithm 5. Identification of all minimal-cover sets of rules for the lower approximation of upward union of classes $\underline{P}(Cl_h^{\geqslant})$.

Input $\mathcal{R}_{all}^{\underline{P}(Cl_h^{\geqslant})}$: all minimal rules covering reference alternatives in $\underline{P}(Cl_h^{\geqslant})$

$\mathcal{R}_{a_i^*}^{\underline{P}(Cl_h^{\geqslant})}$: the subset of all rules covering a reference alternative $a_i^* \in A^R$, i.e.:

$\mathcal{R}_{a_i^*}^{\underline{P}(Cl_h^{\geqslant})} = \{r_i \in \mathcal{R}_{all}^{\underline{P}(Cl_h^{\geqslant})},\ \text{such that}\ a_i^* \in sup(r_i)\}$.

v_i: a binary variable associated with each rule $r_i \in \mathcal{R}_{all}^{\underline{P}(Cl_h^{\geqslant})}$

TS: the current set of constraints.

Output $\mathcal{R}_{mrc}^{\underline{P}(Cl_h^{\geqslant})} = \{\mathcal{R}_1^{\underline{P}(Cl_h^{\geqslant})}, \mathcal{R}_2^{\underline{P}(Cl_h^{\geqslant})}, \ldots, \mathcal{R}_{t_{\underline{P}(Cl_h^{\geqslant})}}^{\underline{P}(Cl_h^{\geqslant})}\}$: the set of all minimal-cover sets of rules for $\underline{P}(Cl_h^{\geqslant})$

1: t = 1.
2: $\mathcal{R}_{mrc}^{\underline{P}(Cl_h^{\geqslant})} = \emptyset$.
3: $TS = \emptyset$.
4: **for** $r_k \in \mathcal{R}_{all}^{\underline{P}(Cl_h^{\geqslant})}$ **do**
5: Add the following constraint to $TS : v_k \in \{0, 1\}$.
6: **end for**
7: **for** $a_i^* \in A^R$ **do**
8: Add the following constraint to $TS : \sum_{r_k \in \mathcal{R}_{a_i^*}^{\underline{P}(Cl_h^{\geqslant})}} v_k \geqslant 1$.
9: **end for**
10: Test the feasibility of TS.
11: **if** feasible **then**
12: Solve the following ILP:
Minimize : $f_t = \sum_{r_k \in \mathcal{R}_{all}^{\underline{P}(Cl_h^{\geqslant})}} v_k$, subject to TS.
13: $\mathcal{R}_t^{\underline{P}(Cl_h^{\geqslant})} = \{r_k \in \mathcal{R}_{all}^{\underline{P}(Cl_h^{\geqslant})},\ \text{such that}\ v_k^* = 1\}$,
where f_t^* is the optimal value of f_t and v_k^* are the values of the binary variables at the corresponding optimum found.
14: $\mathcal{R}_{mrc}^{\underline{P}(Cl_h^{\geqslant})} = \mathcal{R}_{mrc}^{\underline{P}(Cl_h^{\geqslant})} \cup \mathcal{R}_k^{\underline{P}(Cl_h^{\geqslant})}$.
15: Add the following constraint to $TS : \sum_{r_k \in \mathcal{R}_t^{\underline{P}(Cl_h^{\geqslant})}} v_k \leqslant f_t^* - 1$.
16: $t = t + 1$.
17: Go to **10**.
18: **end if**

Illustrative example (part 4): all compatible sets of rules

Using rules from Table 3 as an input for Algorithm 5, we generate all MC sets of rules for the lower approximation of each class union. These are presented in Table 4. For example, there are two MC sets of rules for reference alternatives in $\underline{P}(Cl_1^{\leqslant})$ and $\underline{P}(Cl_2^{\geqslant})$, and eighteen different ways of covering all reference alternatives in $\underline{P}(Cl_4^{\geqslant})$. Let us note that some minimal compatible rules from Table 3 are not used in any MC set of rule (e.g., $r_{\leqslant 1}^1$ or $r_{\leqslant 2}^3$). Overall, there are 38 minimal rule covers for different class unions, out of which 3, 17, 16, and 2 consist of 1, 2, 3, or 4 rules, respectively. Combination of these minimal rule covers as indicated by Eq. (12) leads to 8640 minimal sets of minimal rules $\mathcal{R}^{A^R}$ which reproduce the assignment examples provided by the DM. The number of rules in each of these sets ranges from 10 to 18.

5. Sorting method

The sorting method we propose aims at following the recommendation indicated by the decision rules as faithfully as possible. In fact, each rule r in $\mathcal{R} \in \mathcal{R}^{A^R}$, which covers $a \in A$ suggests a union of classes. In terms of the whole set of rules $\mathcal{R}$ it is reasonable to indicate the intersection of all these unions as the recommendation suggested for a. This method is formulated as Definition 5.1.

Definition 5.1 (*Sorting method using a minimal-cover set of rules $\mathcal{R} \in \mathcal{R}^{A^R}$*). Let us denote by l (u) the lowest (highest) class of the intersection of suggested unions of all "at least" $D_{\geqslant}$ ("at most" $D_{\leqslant}$) decision rules in $\mathcal{R}$ covering a. If l and/or u are undefined or $l \leqslant u$, then a sorting procedure driven by a compatible set of rules $\mathcal{R}$ assigns an alternative $a \in A$ to an interval of classes $[Cl_{L^{\mathcal{R}}(a)}, Cl_{R^{\mathcal{R}}(a)}]$ such that:

$$L^{\mathcal{R}}(a) = Max\{\{1\}, l\}, \tag{13}$$
$$R^{\mathcal{R}}(a) = Min\{\{p\}, u\}. \tag{14}$$

In case of inconsistency (i.e., if $u < l$), a might either be left without recommendation (i.e., the procedure indicates an empty set of classes), or it might be assigned to an interval of classes $[Cl_{L^{\mathcal{R}}(a)=u}, Cl_{R^{\mathcal{R}}(a)=l}]$.

In general, only one of the following three situations can occur when matching alternative a to a set of decision rules $\mathcal{R}$:

1. If no rule covers a, it is assigned to the whole interval of decision classes $[Cl_1, Cl_p]$.
2. If exactly one decision rule covers a, depending on the type of the covering rule, $D_{\leqslant}$ or $D_{\geqslant}$, a is assigned to $[Cl_1, Cl_u]$ or $[Cl_l, Cl_p]$, respectively.
3. If several rules cover a, then in case all of them are of type $D_{\leqslant}$ or $D_{\geqslant}$, a is assigned to $[Cl_1, Cl_u]$ or $[Cl_l, Cl_p]$, respectively. If a is covered by both types of rules, in case $l \leqslant u$, a is assigned to $[Cl_l, Cl_u]$; otherwise, depending on the way of dealing with contradictory information a is assigned to $[Cl_u, Cl_l]$ or left without recommendation explicitly indicating the inconsistency.

6. Possible and necessary assignments

Given a set A^R of assignment examples and a corresponding set $\mathcal{R}^{A^R}$ of compatible MC sets or rules, for each alternative $a \in A$, the possible assignment $C_P(a)$ is defined as the set of indices of classes Cl_h for which there exists at least one compatible set of rules assigning a to Cl_h, and the necessary assignment $C_N(a)$ as the set of indices of classes Cl_h for which all compatible sets of rules assign a to Cl_h. That is, the necessary and possible assignments are:

$$C_P(a) = \{h \in H : \exists \mathcal{R} \in \mathcal{R}^{A^R},\ L^{\mathcal{R}}(a) \leqslant h \leqslant R^{\mathcal{R}}(a)\} = \bigcup_{\mathcal{R} \in \mathcal{R}^{A^R}} [L^{\mathcal{R}}(a), R^{\mathcal{R}}(a)], \tag{15}$$

$$C_N(a) = \{h \in H : \forall \mathcal{R} \in \mathcal{R}^{A^R},\ L^{\mathcal{R}}(a) \leqslant h \leqslant R^{\mathcal{R}}(a)\} = \bigcap_{\mathcal{R} \in \mathcal{R}^{A^R}} [L^{\mathcal{R}}(a), R^{\mathcal{R}}(a)]. \tag{16}$$

Illustrative example (part 5): possible and necessary assignments

Let us examine the variety of the recommendation proposed by all compatible minimal sets of rules for the 46 non-reference alternatives provided in Table 1. We will apply a sorting method discussed in Section 5 referring to a variant which suggests non-empty assignment $[Cl_u, Cl_l]$ in case of inconsistency. We skip the discussion on the recommendation for the 23 reference alternatives, because in case of precise and consistent assignment examples, both the necessary and possible results agree with the assignment of the DM. The possible and necessary assignments for the non-references alternatives are presented in Tables 5 and 6, respectively.

There are 19 non-reference alternatives which are possibly assigned to just a single class, 25 alternatives possibly assigned to two consecutive classes (i.e., $Cl_1 - Cl_2$ or $Cl_2 - Cl_3$ or $Cl_3 - Cl_4$), and 2 alternatives with a possible assignment of three classes ($Cl_2 - Cl_4$). Let us remind that there are two sources of imprecision when it comes to the suggested recommendation. Firstly, a single set of rules may indicate an imprecise assignment for an alternative. Secondly, different set of rules may suggest different class ranges for the same alternative. For the analyzed problem, the average width of the range of possible classes is 1.62 with the greatest hesitation between classes Cl_3 and Cl_4.

When it comes to the necessary assignments, there are 26 alternatives for which it is not empty. This includes 19 alternatives for which the possible assignment is precise, and another 7 alternatives for which the possible assignment is imprecise, but each compatible minimal set of rules indicates a recommendation including the same class (either precisely or imprecisely).

Since all compatible sets of rules are known, for each range of contiguous classes $[Cl_{h_L}, Cl_{h_L+1}, \ldots, Cl_{h_R}]$, with $1 \leqslant h_L \leqslant h_R \leqslant p$, we can define class range acceptability index $CAI(a, [h_L, h_R])$ as the share of compatible sets of rules $\mathcal{R} \in \mathcal{R}^{A^R}$ that assign alternative a precisely to the range of classes $[Cl_{h_L}, Cl_{h_L+1}, \ldots, Cl_{h_R}]$ (i.e., $L^{\mathcal{R}}(a) = h_L$ and $R^{\mathcal{R}}(a) = h_R$), i.e.:

Table 5
Possible assignments for non-reference alternatives.

Class range	Assigned zones
Cl_4–Cl_4	a_2, a_{16}, a_{17}, a_{53}, a_{88}, a_{90}
Cl_3–Cl_4	a_8, a_{14}, a_{20}, a_{21}, a_{24}, a_{35}, a_{36}, a_{41}, a_{53}, a_{89}, a_{93}, a_{95}, a_{96}, a_{97}, a_{99}
Cl_3–Cl_3	a_5, a_9, a_{10}, a_{22}, a_{29}, a_{83}, a_{100}
Cl_2–Cl_4	a_{92}, a_{94}
Cl_2–Cl_3	a_{28}, a_{42}, a_{44}, a_{56}, a_{85}, a_{86}, a_{87}
Cl_2–Cl_2	a_{25}, a_{26}, a_{37}, a_{43}, a_{106}, a_{107}
Cl_1–Cl_2	a_{43}, a_{75}, a_{112}
Cl_1–Cl_1	a_{76}

Table 6
Necessary assignments for non-reference alternatives.

Class	Assigned zones
Cl_4	a_2, a_{14}, a_{16}, a_{17}, a_{35}, a_{53}, a_{88}, a_{90}, a_{97}
Cl_3	a_5, a_9, a_{10}, a_{22}, a_{29}, a_{36}, a_{41}, a_{83}, a_{87}, a_{100}
Cl_2	a_{25}, a_{26}, a_{37}, a_{43}, a_{106}, a_{107}
Cl_1	a_{76}

$$CAI(a,[h_L,h_R]) = \sum_{\mathcal{R}\in\mathcal{R}^{A^R}} m(\mathcal{R},a,[h_L,h_R])/|\mathcal{R}^{A^R}|, \tag{17}$$

where $m(\mathcal{R},a,[h_L,h_R])$ is the class range membership function:

$$m(\mathcal{R},a,[h_L,h_R]) = \begin{cases} 1, & \text{if } L^{\mathcal{R}}(a) = h_L \text{ and } R^{\mathcal{R}}(a) = h_R, \\ 0, & \text{otherwise}. \end{cases} \tag{18}$$

The class acceptability index can be interpreted as an estimate of a probability of membership to the particular class interval [31].

We can also compute the share of $\mathcal{R}\in\mathcal{R}^{A^R}$ for which Cl_h is within $[Cl_{L^{\mathcal{R}}(a)},\ldots,Cl_{R^{\mathcal{R}}(a)}]$, i.e. the share of sets of rules that either precisely or imprecisely assign a to Cl_h. Following [31], let us call such a share the cumulative class acceptability index $CuCAI(a,h)$:

$$CuCAI(a,h) = \sum_{[h_L,h_R]:h\in[h_L,h_R]} CAI(a,[h_L,h_R]). \tag{19}$$

The following propositions indicating interdependencies between the possible and necessary assignments and *CuCAI*s are true.

Proposition 6.1. *If $CuCAI(a,h) > 0$, then $h \in C_P(a)$ and if $CuCAI(a,h) = 0$, then $h \notin C_P(a)$;*

Proposition 6.2. *If $CuCAI(a,h) = 1$, then $h \in C_N(a)$ and if $CuCAI(a,h) < 1$, then $h \notin C_N(a)$.*

Note that apart from considering certain decision rules generated from the lower approximations of class unions, we may additionally take into account possible decision rules based on the upper approximations of class unions. Then, two levels of certainty for the results should be considered. The first level is related to the necessary (N) and possible (P) consequences of preference information provided by the DM, i.e. application of certain and possible decision rules, respectively. The other level refers to the subset of compatible sets of (certain or possible) rules confirming the specific outcome. In this way, we would arrive at four types of results:

- necessary–necessary (N–N), i.e. results which are confirmed for all compatible sets of certain rules;
- necessary–possible (N–P), i.e. results which are confirmed for at least one compatible set of certain rules;
- possible–necessary (P–N), i.e. results which are confirmed for all compatible sets of possible rules;
- possible–possible (P–P), i.e. results which are confirmed for at least one compatible set of possible rules.

Illustrative example (part 6): class acceptability indices

Let us examine the way the class acceptability indices enrich the analysis of the necessary and possible assignments for this particular problem. Obviously, the 19 zones that are possibly assigned to a single class have the class *CAI* 100% and *CAI*s of other classes zero (see Table 7, e.g., a_2, a_5, a_{25}, or a_{76}). For many alternatives that are possibly assigned to at least two consecutive classes, we can indicate a single recommendation suggested by the majority of compatible MC sets of rules (e.g., $CAI(a_{14},Cl_4) = 88.89\%, CAI(a_{41},Cl_3) = 75.00\%,\ CAI(a_{95},Cl_3) = 62.50\%$). Note that since a single set of rules may provide imprecise assignment, the prevailing recommendation does not have to be precise (e.g., $CAI(a_{53},Cl_3 - Cl_4) = 51.39\%$, $CAI(a_{56},Cl_2 - Cl_3) = 53.33\%, CAI(a_{75},Cl_1 - Cl_2) = 50.00\%$).

For other zones, an analysis of *CAI*s allows narrowing down the range of most probable recommendations. For example, for over 95% of the compatible MC sets of rules, a_{96} is assigned to class range $Cl_3 - Cl_4$ or class Cl_4, whereas only about 4% assign it precisely to class Cl_3. Furthermore, for over 93% of the compatible MC sets of rules, a_{28} is assigned to $Cl_2 - Cl_3$ or Cl_3, whereas only less than 7% of the sets of rules admit it to Cl_2.

The analysis can be further enhanced by considering *CuCAI*s which are also presented in Table 7. Obviously, the 26 alternatives with the non-empty necessary assignment have the class *CuCAI* 100%. Let us note that 7 of them (a_{14}, $a_{35}, a_{36}, a_{41}, a_{43}, a_{87}$, and a_{97}) have *CuCAI* greater than zero also for some other classes, which implies that the possible assignment is wider than the necessary one. While there are alternatives whose *CuCAIs* are balanced (e.g., a_{53}, a_{86}, or a_{112}), there is also a wide spectrum of alternatives for which the vast majority of compatible sets of rules assign them to a certain class. For example, a_{96}, a_{24}, and a_{21} are possibly assigned to Cl_4 with more than 91% of the compatible sets of rules (resulting from their possible assignments to $[Cl_3, Cl_4]$ or Cl_4), whereas other 7 alternatives (e.g., a_8 or a_{89}) are possibly assigned to Cl_3 with more than 90% of the compatible sets of rules.

7. Selection of a representative minimal-cover set of rules

In this section, we introduce the concept of a representative MC set of rules for multiple criteria sorting. We have already considered an analogous problem in case of value function and outranking relation preference models [27–29]. We propose some pre-defined procedures which aim at selecting a single set of rules representing the whole set of compatible sets of rules. This set of rules builds on results of ROR, making use of the class acceptability indices. Representativeness of the selected set of rules is understood in the sense of the robustness concern. The representative set of rules is expected to produce a robust recommendation with respect to the non-univocal preference model stemming from the input preference information. Thus, we wish to indicate a single preference model instance whose recommendation is confirmed by as many other compatible instances as possible. Obviously, this general idea may be implemented in several ways, e.g.:

Table 7
Class acceptability indices (*CAIs*), cumulative class acceptability indices (*CuCAIs*), and assignments by the representative sets of rules.

Zone	*CAIs*								*CuCAIs*				Repr. assign.	
	1–1	1–2	2–2	2–3	2–4	3–3	3–4	4–4	1	2	3	4	C^{REP}_{SUM}	C^{REP}_{MIN}
a_2	–	–	–	–	–	–	–	100.0	–	–	–	100.0	4–4	4–4
a_5	–	–	–	–	–	100.0	–	–	–	–	100.0	–	3–3	3–3
a_8	–	–	–	–	–	45.83	44.44	9.73	–	–	90.28	54.17	3–3	3–3
a_9	–	–	–	–	–	100.0	–	–	–	–	100.0	–	3–3	3–3
a_{10}	–	–	–	–	–	100.0	–	–	–	–	100.0	–	3–3	3–3
a_{14}	–	–	–	–	–	–	11.11	88.89	–	–	11.11	100.0	4–4	4–4
a_{16}	–	–	–	–	–	–	–	100.0	–	–	–	100.0	4–4	4–4
a_{17}	–	–	–	–	–	–	–	100.0	–	–	–	100.0	4–4	4–4
a_{20}	–	–	–	–	–	50.00	41.67	8.33	–	–	91.67	50.00	3–4	3–4
a_{21}	–	–	–	–	–	8.33	50.00	41.67	–	–	58.33	91.67	4–4	4–4
a_{22}	–	–	–	–	–	100.0	–	–	–	–	100.0	–	3–3	3–3
a_{24}	–	–	–	–	–	5.55	50.00	44.44	–	–	55.55	94.44	4–4	4–4
a_{25}	–	–	100.0	–	–	–	–	–	–	100.0	–	–	2–2	2–2
a_{26}	–	–	100.0	–	–	–	–	–	–	100.0	–	–	2–2	2–2
a_{28}	–	–	6.67	40.00	–	53.33	–	–	–	46.67	93.33	–	3–3	3–3
a_{29}	–	–	–	–	–	100.0	–	–	–	–	100.0	–	3–3	3–3
a_{35}	–	–	–	–	–	–	16.67	83.33	–	–	16.67	100.0	4–4	4–4
a_{36}	–	–	–	–	–	62.50	37.50	–	–	–	100.0	37.50	3–3	3–3
a_{37}	–	–	100.0	–	–	–	–	–	–	100.0	–	–	2–2	2–2
a_{41}	–	–	–	–	–	75.00	25.00	–	–	–	100.0	25.00	3–3	3–3
a_{42}	–	–	6.67	40.00	–	53.33	–	–	–	46.67	93.33	–	3–3	3–3
a_{43}	–	50.00	50.00	–	–	–	–	–	50.00	100.0	–	–	1–2	1–2
a_{44}	–	–	6.67	40.00	–	53.33	–	–	–	46.67	93.33	–	3–3	3–3
a_{52}	–	–	–	–	–	–	–	100.0	–	–	–	100.0	4–4	4–4
a_{53}	–	–	–	–	–	27.78	51.39	20.83	–	–	79.17	72.22	3–4	3–3
a_{56}	–	–	26.67	53.33	–	20.00	–	–	–	80.00	73.33	–	2–2	2–2
a_{75}	25.00	50.00	25.00	–	–	–	–	–	75.00	75.00	–	–	1–2	1–2
a_{76}	100.0	–	–	–	–	–	–	–	100.0	–	–	–	1–1	1–1
a_{83}	–	–	–	–	–	100.0	–	–	–	–	100.0	–	3–3	3–3
a_{85}	–	–	26.67	53.33	–	20.00	–	–	–	80.00	73.33	–	2–2	2–2
a_{86}	–	–	26.67	53.33	–	20.00	–	–	–	80.00	73.33	–	2–2	2–2
a_{87}	–	–	–	40.00	–	60.00	–	–	–	40.00	100.0	–	2–3	2–3
a_{88}	–	–	–	–	–	–	–	100.0	–	–	–	100.0	4–4	4–4
a_{89}	–	–	6.67	40.00	–	53.33	–	–	–	46.67	93.33	–	3–3	3–3
a_{90}	–	–	–	–	–	–	–	100.0	–	–	–	100.0	4–4	4–4
a_{92}	–	–	22.22	20.37	35.18	3.70	11.11	7.41	–	77.77	70.37	53.70	2–3	2–3
a_{93}	–	–	–	–	–	13.89	50.00	36.11	–	–	63.89	86.11	3–3	3–3
a_{94}	–	–	28.88	33.70	18.52	9.63	9.26	–	–	81.11	71.11	27.77	2–3	2–3
a_{95}	–	–	–	–	–	62.50	33.33	4.61	–	–	95.83	37.50	3–3	3–3
a_{96}	–	–	–	–	–	4.17	33.33	62.50	–	–	37.50	95.83	4–4	4–4
a_{97}	–	–	–	–	–	–	11.11	88.89	–	–	11.11	100.0	4–4	4–4
a_{99}	–	–	–	–	–	33.33	52.27	13.89	–	–	86.11	66.67	3–4	4–4
a_{100}	–	–	–	–	–	100.0	–	–	–	–	100.0	–	3–3	3–3
a_{106}	–	–	100.0	–	–	–	–	–	–	100.0	–	–	2–2	2–2
a_{107}	–	–	100.0	–	–	–	–	–	–	100.0	–	–	2–2	2–2
a_{112}	25.00	50.00	25.00	–	–	–	–	–	75.00	75.0	–	–	1–2	1–2

- a representative MC set of rules $\mathcal{R}_{rep}$ may be interpreted as a single compatible set of rules $\mathcal{R} \in \mathcal{R}^{A^R}$ for which a minimal acceptability index for a class range indicated by $\mathcal{R}$ for any alternative $a \in A$, is maximal among all MC compatible sets of rules $\mathcal{R}^{A^R}$, i.e.:

$$\mathcal{R}_{rep} = \mathcal{R} \in \mathcal{R}^{A^R}, \quad \text{for which } \min_{a \in A} CAI(a, [L^{\mathcal{R}}(a), R^{\mathcal{R}}(a)]) \text{ is maximal in } \mathcal{R}^{A^R}; \tag{20}$$

- alternatively, $\mathcal{R}_{rep}$ may be interpreted as $\mathcal{R} \in \mathcal{R}^{A^R}$ for which a sum of acceptability indices for class ranges indicated by $\mathcal{R}$ for all alternatives $a \in A$, is maximal in $\mathcal{R}^{A^R}$, i.e.:

$$\mathcal{R}_{rep} = \mathcal{R} \in \mathcal{R}^{A^R}, \quad \text{for which } \sum_{a \in A} CAI(a, [L^{\mathcal{R}}(a), R^{\mathcal{R}}(a)]) \text{ is maximal in } \mathcal{R}^{A^R}. \tag{21}$$

Illustrative example (part 7): representative minimal-cover sets of rules

The representative MC set of rules which maximizes a sum of acceptability indices for class ranges is presented in Table 8. The set of rules which maximizes a minimal acceptability index differs just by a single rule, replacing: "if ($g_1 \leqslant 66.4$) and ($g_2 \leqslant 2$) and ($g_3 \leqslant 4$) then $a \in Cl_4^{\geqslant}$" with "if ($g_1 \leqslant 99.3$) and ($g_3 \leqslant 4$) and ($g_4 \leqslant 3$) then $a \in Cl_4^{\geqslant}$".

Results obtained with the use of a representative set of rules can be analyzed in the context of final outcomes of ROR. For purpose of illustration, we provide recommendation suggested by both representative sets of rules in Table 7 (see columns C_{SUM}^{REP} and C_{MIN}^{REP}, respectively), but it is advisable that the DM, assisted by an analyst, chooses only one of them. Such analysis is useful because, in general, these representative assignments are more precise than possible assignments which can be very wide (even equal to the whole range of classes), and at the same time they are more general than necessary assignments which, on the other hand, can be empty. Thus, introducing the concept of a representative set of rules, we extend ROR in its capacity of explaining the final output in terms of a compatible model which can be displayed to the DM. For any user, the analysis of a single, representative set of rules is surely less abstract than that of the whole set of compatible sets or rules.

8. Decisive rules in terms of the provided recommendation

Considering a compatible set of rules $\mathcal{R} \in \mathcal{R}^{A^R}$, only some of the rules covering $a \in A$ directly influence the assignment $[L^{\mathcal{R}}(a), R^{\mathcal{R}}(a)]$ indicated by $\mathcal{R}$. Precisely, these rules indicate a class union equal to the intersection of unions suggested for a by all "at least" $D_{\geqslant}$- or all "at most" $D_{\leqslant}$-decision rules. Let us call such rules, being of a particular interest to the DM, *decisive* in terms of the recommendation provided by $\mathcal{R}$ for an alternative a. In this case, a function $dec(\mathcal{R}, a, r_i)$ indicating such decisiveness of a rule r_i is equal to 1; otherwise, it is equal to 0. From the DM's point of view, the most useful rules when analyzing and explaining the recommendation provided for a by all compatible sets of rules are these being decisive for the greatest share of compatible sets of rules $\mathcal{R} \in \mathcal{R}^{A^R}$, i.e. for each $a \in A$:

$$r_i \in \mathcal{R}_{all}^{A^R} : dec(\mathcal{R}^{A^R}, a, r_i) = \sum_{\mathcal{R} \in \mathcal{R}^{A^R}} dec(\mathcal{R}, a, r_i) / \mid \mathcal{R}^{A^R} \mid \text{ is maximal in } \mathcal{R}_{all}^{A^R}. \tag{22}$$

Let us call $dec(\mathcal{R}^{A^R}, a, r_i)$ a decisiveness factor for alternative a, rule $r_i \in \mathcal{R}_{all}^{A^R}$, and a set of compatible sets of rules $\mathcal{R}^{A^R}$. The above criterion may used separately in case of "at least" $D_{\geqslant}$- or "at most" $D_{\leqslant}$-rules.

Illustrative example (part 8): decisive rules

In Table 9, we present the decisive rules for two exemplary non-reference alternatives a_{25} and a_{36} along with their decisiveness factors. The previous alternative (a_{25}) is necessarily assigned to class Cl_2 by all compatible sets of rules $\mathcal{R}^{A^R}$.

Table 8
Representative MC set of rules selected according to (21).

if ($g_1 \leqslant 163.1$) then $a \in Cl_2^{\geqslant}$
if ($g_2 \leqslant 5$) then $a \in Cl_2^{\geqslant}$
if ($g_3 \leqslant 4$) then $a \in Cl_3^{\geqslant}$
if ($g_1 \leqslant 99.3$) and ($g_2 \leqslant 5$) then $a \in Cl_3^{\geqslant}$
if ($g_3 \leqslant 1$) and ($g_4 \leqslant 3$) then $a \in Cl_4^{\geqslant}$
if ($g_2 \leqslant 3$) and ($g_4 \leqslant 1$) then $a \in Cl_4^{\geqslant}$
if ($g_1 \leqslant 66.4$) and ($g_2 \leqslant 2$) and ($g_3 \leqslant 4$) then $a \in Cl_4^{\geqslant}$
if ($g_5 \geqslant 5$) then $a \in Cl_1^{\leqslant}$
if ($g_5 \geqslant 2$) then $a \in Cl_2^{\leqslant}$
if ($g_1 \geqslant 108.2$) and ($g_3 \geqslant 6$) then $a \in Cl_2^{\leqslant}$
if ($g_4 \geqslant 4$) then $a \in Cl_3^{\leqslant}$
if ($g_3 \geqslant 6$) and ($g_4 \geqslant 3$) then $a \in Cl_3^{\leqslant}$
if ($g_2 \geqslant 5$) and ($g_3 \geqslant 2$) then $a \in Cl_3^{\leqslant}$

On the one hand, the indication of Cl_2 as the lower class is caused by $r^2_{\geqslant 2}$ or $r^3_{\geqslant 2}$ in 50% of the compatible sets of rules. On the other hand, $r^1_{\leqslant 2}$ and $r^4_{\leqslant 2}$ indicate Cl_2 as the upper class for a_{25} in 67% of the compatible sets of rules. When it comes to a_{36} which is assigned to a class range $Cl_3 - Cl_4$ by 62.50% and to a class Cl_4 by 37.50% compatible sets of rules, the most decisive rules are $r^3_{\geqslant 3}$ for the lower class and $r^2_{\leqslant 3}$ for the upper class.

9. Discussion

In this paper, we presented a new approach to multiple criteria sorting problems deriving from Dominance-based Rough Set Approach. In order to specify the sorting model, the Decision Maker is asked to provide desired assignments for reference alternatives which are relatively well-known to her/him. The underlying preference model is a set of all compatible minimal sets of rules. Then, the method provides for each alternative two assignments: the necessary assignment corresponds to the interval of classes to which alternative can be assigned considering all compatible sets of rules simultaneously, while the possible assignment corresponds to the range of classes to which alternative is assigned by at least one compatible set of rules. Additionally, for each alternative we provide class acceptability indices and a representative assignment resulting from the use of a single minimal set of rules which is representative for all other compatible sets of rules in terms of robustness preoccupation. The introduced approach was illustrated on an example of assigning land zones to different classes of risk.

The discussed approach remains valid for different types of holistic preference information. In particular, apart from assignment examples we may use the desired class cardinalities [30] (e.g., "Cl_1 should contain at least 15 alternatives" or "Cl_2 should contain at most 20% of alternatives") or assignment-based pairwise comparisons [25] (e.g., "a should be assigned to a class better than b" or "c is better than d by at least two classes"). Then, the sets of rules which are compatible with the assignment examples, not being consistent with other types of preference information would be simply eliminated from the set of all compatible sets of rules.

The introduced method is also easily extendable to Variable Consistency Dominance-based Rough Set Approach (VC-DRSA) [4,18,20,43]. VC-DRSA produces more general (extended) lower approximations than those computed by Dominance-based Rough Set Approach, (i.e., lower approximations that are supersets of those computed by DRSA) containing alternatives characterized by a strong but not necessarily certain relation with approximated sets.

The method has been integrated with a java Rough Set (jRS) library (see, e.g., [2]) and lpSolve which is used for solving the Integer Linear Programming problems. The execution time for our illustrative problem on Intel Atom CPU D325 1.80 GHz with 4 GB RAM was less than 2 s. This time is highly satisfying for typical Multiple Criteria Decision Aiding applications. In fact, multiple criteria problems considered in Operations Research and Management Science (OR/MS) usually involve several dozens or at most few hundreds of alternatives [49]. Thus, it is not realistic to assume the DM would be willing to provide more than few tens of assignment examples when dealing with such a set of alternatives. Nevertheless, one of the future research directions concerns parallel implementation of the proposed algorithms with a Map-Reduce framework.

In the view of the above remark, let us add that for typical MCDA applications Robust Ordinal Regression for Dominance-based Rough Set Approach works faster than ROR implemented for value- or outranking-based methods. It is the case, even though the latter approaches do not require explicit generation of all compatible preference model instance, but rather exploit the set of mathematical constraints delimiting all compatible value functions or outranking models, respectively. However, in case of value-drive sorting procedures, to determine the possible (necessary) assignment for all alternatives one needs to solve $h \cdot n$ $(2|A^R| \cdot n)$ Linear Programming (LP) problems with the range of dimension $n \cdot m + |A^R|$ constraints and $n \cdot m$ variables. Although this is not a burden for the contemporary solvers, the computational effort increases with the number of alternatives n. On the contrary, in the proposed rule-based approach, the greatest computational effort is

Table 9
Decisive rules for the exemplary non-reference alternatives.

Rule r_i	$dec(\mathcal{R}^{A^R}, a_{25}, r_i)$
Decisive rules for a_{25}	
$r^2_{\geqslant 2}$: if $(g_5 \leqslant 2)$ then $a \in Cl_2^{\geqslant}$	0.50
$r^3_{\geqslant 2}$: if $(g_1 \leqslant 163.1)$ then $a \in Cl_2^{\geqslant}$	0.50
$r^1_{\leqslant 2}$: if $(g_2 \geqslant 7)$ then $a \in Cl_2^{\leqslant}$	0.67
$r^2_{\leqslant 2}$: if $(g_5 \geqslant 2)$ then $a \in Cl_2^{\leqslant}$	0.33
$r^4_{\leqslant 2}$: if $(g_1 \geqslant 108.2)$ and $(g_3 \geqslant 6)$ then $a \in Cl_2^{\leqslant}$	0.67
	$dec(\mathcal{R}^{A^R}, a_{36}, r_i)$
Decisive rules for a_{36}	
$r^3_{\geqslant 3}$: if $(g_2 \leqslant 3)$ then $a \in Cl_3^{\geqslant}$	0.60
$r^5_{\geqslant 3}$: if $(g_1 \leqslant 99.3)$ and $(g_5 \leqslant 1)$ then $a \in Cl_3^{\geqslant}$	0.20
$r^7_{\geqslant 3}$: if $(g_1 \leqslant 99.3)$ and $(g_2 \leqslant 5)$ then $a \in Cl_3^{\geqslant}$	0.20
$r^2_{\leqslant 3}$: if $(g_4 \geqslant 4)$ then $a \in Cl_3^{\leqslant}$	0.625
$r^6_{\leqslant 3}$: if $(g_3 \geqslant 6)$ and $(g_4 \geqslant 3)$ then $a \in Cl_3^{\leqslant}$	0.25

related to the generation phase, while once all minimal sets of rules are induced, they are just applied to one alternative after another.

In any case, taking into account the complexity of the method, an approximate approach addressing the existence of multiple MC set of rules may be employed to deal with greater number of assignment examples and/or criteria. For example, a greedy heuristic of sequential covering type, such as DomLEM [19], may be run a certain number of times (e.g., 100) for different randomly generated subsets of criteria (if available, reducts) or with some minor perturbation (e.g., not choosing the best candidate for the elementary condition, but rather the second one). In this way, potentially many MC set of rules can be obtained. Then, the robustness analysis of the recommendation provided by them may be performed as proposed in this paper.

We envisage the following developments of the proposed approach:

- extensions of the method for group decision making problems, multiple criteria ranking and choice, and decision making under uncertainty,
- higher variety of analyzed results, for example, assignment-based preference relations [31] and extreme class cardinalities,
- identification of minimal sets of the provided assignment examples that imply the obtained possible and necessary assignments for each alternative [26],
- application to some real-world sorting problems in environmental management and economy.

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